

Symmetry, reductions and exact solutions of the difference equation $u_{n+2} = au_n/(1 + bu_nu_{n+1})$

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Abstract

We investigate the solutions of the second-order difference equation $u_{n+2} = (au_n)/(1 + bu_nu_{n+1})$ using a group of transformations (Lie symmetries) that leaves the solutions invariant.

Key words: Difference equation; symmetry; reduction; group invariant solutions

1 Introduction

Symmetry methods for differential equations are well-documented and have been extended to difference equations recently [1–4]. The idea consists of finding symmetries of the equations and use them to lower the order of the equation. Once the solutions of the reduced equations are obtained, one can retrieve the solutions of the original equation by using the invariance of the difference equation under the group of transformations or using the similarity variables.

Some authors have studied the solutions of

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_{n-1}x_n}, \quad (1)$$

where a and b are real numbers, by putting some restrictions on a, b and the initial conditions x_{-1} and x_0 . Aloqeili [6] investigated the solutions, the stability properties and semi-cycle behavior of equation (1) when $a = -b = 1/A$ with $A \geq 0$, $x_{-1}x_0 \neq A^j(1 - A)(1 - A^j)$, i.e.,

$$x_{n+1} = \frac{x_{n-1}}{A - x_{n-1}x_n}. \quad (2)$$

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Cinar [7–11] investigated the solutions of

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{1 + x_{n-1}x_n}, x_{n+1} = \frac{x_{n-1}}{-1 + x_{n-1}x_n}, x_{n+1} = \frac{x_{n-1}}{-1 + Bx_{n-1}x_n}, \\ x_{n+1} &= \frac{x_{n-1}}{1 + Ax_{n-1}x_n}, x_{n+1} = \frac{ax_{n-1}}{1 + bx_{n-1}x_n} \end{aligned} \quad (3)$$

with the assumptions that

- x_{-1}, x_0 are positive real numbers
- x_{-1}, x_0 are real numbers such that $x_0x_{-1} \neq 1$
- $B \geq 0$, x_{-1} and x_0 are real numbers such that $Bx_0x_{-1} \neq 1$
- A, x_{-1} and x_0 are non-negative real numbers
- a, b, x_{-1} and x_0 are non-negative real numbers,

respectively.

We aim to obtain the solutions of (1) using its symmetry. We expect our solutions to be more general (with less restrictions on a and b) and in a ‘single form’ (contrarily to what were presented by these authors). In order to use our method we will have to ‘shift’ equation (1) and study the equation

$$u_{n+2} = \frac{au_n}{1 + bu_nu_{n+1}} \quad (4)$$

instead. There should be bijections that map our solutions to their solutions. These bijections (if any) will also be investigated.

1.1 Overview about Lie analysis of difference equations

Let us consider a p th-order difference equation in its general form

$$u_{n+p} = \omega(n, u_n, u_{n+1}, \dots, u_{n+p-1}), \quad (5)$$

for some function ω , and the point transformations

$$\Gamma_\epsilon : (n, u_n) \mapsto (n, u_n + \epsilon Q(n, u_n)), \quad (6)$$

for some continuous function Q which we shall refer to as a characteristic.

Definition 1.1 *The forward shift operator is defined as follows:*

$$S : n \mapsto n + 1, \quad S^i u_n = u_{n+i}. \quad (7)$$

The reader can easily check that Γ_ϵ is a one-parameter Lie group of transformation admitting

$$X = Q \frac{\partial}{\partial u_n} + SQ \frac{\partial}{\partial u_{n+1}} + \cdots + S^{p-1} Q \frac{\partial}{\partial u_{n+p-1}} \quad (8)$$

as an infinitesimal generator. The characteristics $Q = Q(n, u_n, \dots, u_{n+p-1})$ can be found by solving the linearized symmetry condition

$$\mathcal{S}^{(p)} Q - X\omega = 0 \quad (9)$$

whenever (5) holds.

Definition 1.2 *A function V_n is invariant under the Lie group of transformations Γ_ϵ if and only if*

$$X(V_n) = 0. \quad (10)$$

Suppose the characteristic Q is known, the invariant V_n can be found by solving the characteristics equation

$$\frac{du_n}{Q} = \frac{du_{n+1}}{SQ} = \cdots = \frac{du_{n+p-1}}{S^{n+p-1}Q} \left(= \frac{dV_n}{0} \right). \quad (11)$$

The reader can refer to [3, 12] to deepen his knowledge of how to use symmetry methods for difference equations. To the best of our knowledge, there are no packages or computer algebra systems that generate symmetries of difference equations. Often times, the computation becomes cumbersome and some extra ansatz may be needed in order to find the characteristics.

2 Main results

Consider the difference equation (4). Imposing the symmetry condition (9) we get

$$Q(n+2, u_{n+2}) - \frac{au_n^2}{(bu_n u_{n+1} + 1)^2} Q(n+1, u_{n+1}) + \frac{a}{(bu_n u_{n+1} + 1)^2} Q(n, u_n). \quad (12)$$

The latter is an equation containing a function, Q , with different arguments making it difficult to solve. To overcome this, we shall assume that u_{n+1} is a function of n , u_n and ω . We then proceed by differentiating (12) with respect to u_n (keeping ω fixed). This leads to

$$-Q'(n+1, u_{n+1}) + Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n) = 0. \quad (13)$$

By differentiating (13) with respect to u_n (keeping u_{n+1} fixed) we obtain

$$Q''(n, u_n) - \frac{2}{u_n} Q'(n, u_n) + \frac{2}{u_n^2} Q(n, u_n) = 0 \quad (14)$$

whose solution is given by

$$Q(n, u_n) = \alpha(n)u_n^2 + \beta(n)u_n \quad (15)$$

for some functions α and β of n .

The last step will consist of substituting (15) in (12) to get the symmetry given by

$$X = (-1)^n u_n \partial u_n - (-1)^n u_{n+1} \partial u_{n+1}. \quad (16)$$

It is easy to check that the function

$$v_n = u_n u_{n+1} \quad (17)$$

is invariant under X given in (16) and that

$$v_{n+1} = \frac{av_n}{1 + bv_n}. \quad (18)$$

The solution of (18) is given by

$$v_n = \begin{cases} \frac{c_1 \left(\frac{1}{a}\right)^{n-2} - c_1 \left(\frac{1}{a}\right)^{n-1} - b \left(\frac{1}{a}\right)^n + b}{c_1 \left(\frac{1}{a}\right)^{n-2} - c_1 \left(\frac{1}{a}\right)^{n-1} - b \left(\frac{1}{a}\right)^n + b} & \text{if } a \neq 1, \\ \frac{1}{bn + c_0} & \text{if } a = 1 \end{cases} \quad (19)$$

for some constants c_0 and c_1 .

2.1 Case $a = 1$

When $a = 1$ equation (4) becomes

$$u_{n+2} = \frac{u_n}{1 + bu_n u_{n+1}}. \quad (20)$$

In this case we are saying that the solution of (18) is given by

$$v_n = \frac{1}{bn + c_0}. \quad (21)$$

Invoking (19), we have that

$$u_{n+1} = \frac{1}{(bn + c_0)u_n}. \quad (22)$$

Note: The order of equation (20) has been reduced by one.

The solution of (22) given by

$$u_n = \exp \left((-1)^{n-1} c_2 + (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{-k_1} \ln |v_{k_1}| \right) \quad (23)$$

$$= \exp \left((-1)^{n-1} c_2 + (-1)^{n-1} \sum_{k_1=0}^{n-1} -(-1)^{-k_1} \ln |c_0 + bk_1| \right), \quad (24)$$

where c_2 is an arbitrary constant, is also the solution of (20). It has to be noted that $c_0 = \frac{1}{u_0 u_1}$ and $c_2 = \ln \left| \frac{1}{u_0} \right|$ in this case. Therefore, the most general solution of (20) is given by

$$u_n = \exp \left((-1)^{n-1} \ln \left| \frac{1}{u_0} \right| + (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{-k_1} \ln \left| \frac{u_0 u_1}{1 + bu_0 u_1 k_1} \right| \right), \quad (25a)$$

$$= \exp \left((-1)^{n-1} \ln |u_1| + (-1)^{n-1} \sum_{k_1=1}^{n-1} (-1)^{-k_1} \ln \left| \frac{u_0 u_1}{1 + bu_0 u_1 k_1} \right| \right), \quad (25b)$$

$$\text{with } -1/(bu_0 u_1) \notin \{1, 2, \dots, n-1\}. \quad (25c)$$

Remark 2.1 *The solution (25a) can be split into.*

$$u_n = \begin{cases} u_0 \frac{\prod_{t=0}^{n/2} [1+2tbu_0u_1]}{\prod_{t=0}^{n/2} [1+(2t+1)bu_0u_1]} & n \text{ even} \\ u_1 \frac{\prod_{t=0}^{\frac{n-1}{2}-1} [1+(2t+1)bu_0u_1]}{\prod_{t=0}^{\frac{n-1}{2}} [1+2tbu_0u_1]} & n \text{ odd} \end{cases} \quad (26)$$

- If we let $x_n = u_{n+1}$, $u_0 = k$, $u_1 = h$, $a = 1$ and $b = A$, we get the result

$$x_n = \begin{cases} k \frac{\prod_{i=0}^{\frac{n+1}{2}-1} [2A h k i + 1]}{\prod_{i=0}^{\frac{n+1}{2}-1} [(2i+1)A h k + 1]} & n \text{ odd} \\ h \frac{\prod_{i=0}^{\frac{n}{2}-1} [1+(2i+1)A h k]}{\prod_{i=0}^{\frac{n}{2}} [2iA h k + 1]} & n \text{ even} \end{cases} \quad (27)$$

obtained by C. Cinar in [10] for

$$x_{n+1} = \frac{x_{n-1}}{1 + A x_n x_{n-1}} \quad (28)$$

and his restriction (x_{-1} , x_0 and A are positive real numbers) is a special case of our restriction given in (25c), that is, $-1/(A x_{-1} x_0) \notin \{1, 2, \dots, n-1\}$ in this case.

- If we let $x_n = u_{n+1}$, $u_0 = k$, $u_1 = h$, $a = 1$, $b = 1$, we get the result

$$x_n = \begin{cases} k \frac{\prod_{i=0}^{\frac{n+1}{2}-1} [2h k i + 1]}{\prod_{i=0}^{\frac{n+1}{2}-1} [(2i+1)h k + 1]} & n \text{ odd} \\ h \frac{\prod_{i=0}^{\frac{n}{2}-1} [1+(2i+1)h k]}{\prod_{i=0}^{\frac{n}{2}} [2i h k + 1]} & n \text{ even} \end{cases} \quad (29)$$

obtained by C. Cinar in [7] for

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}} \quad (30)$$

and his restriction (x_{-1} and x_0 are positive real numbers) is a special case of our restriction given in (25c), that is, $-1/(x_{-1} x_0) \notin \{1, 2, \dots, n-1\}$ in this case.

2.2 Case $a \neq 1$

We mentioned earlier that the solution of (18) when $a \neq 1$ is given by

$$v_n = \frac{a-1}{c_1 \left(\frac{1}{a}\right)^{n-2} - c_1 \left(\frac{1}{a}\right)^{n-1} - b \left(\frac{1}{a}\right)^n + b}. \quad (31)$$

Here, $c_1 = 1/au_0u_1$ and the above equation simplifies to

$$\begin{aligned} v_n &= \frac{(a-1)u_0u_1a^n}{a-1+bu_0u_1(a^n-1)} \\ &= \frac{u_0u_1a^n}{1+bu_0u_1(\sum_{i=0}^{n-1} a^i)}. \end{aligned} \quad (32)$$

Thank to (17) we have $u_{n+1} = v_n/u_n$ and then,

$$\begin{aligned} u_n &= \exp \left[(-1)^{n-1} \ln \left| \frac{1}{u_0} \right| + (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{-k_1} \ln |v_{k_1}| \right], \\ &= \exp \left[(-1)^{n-1} \left(\ln |u_1| + \sum_{k_1=1}^{n-1} (-1)^{-k_1} \ln \left| \frac{u_0u_1a^{k_1}}{1+bu_0u_1(\sum_{i=0}^{k_1-1} a^i)} \right| \right) \right] \end{aligned} \quad (33)$$

$$\text{with } -1/(bu_0u_1) \notin \{1, 1+a, \dots, \sum_{i=0}^{n-2} a^i\}. \quad (34)$$

Remark 2.2 *The solution (33) can be split into*

$$u_n = \begin{cases} u_0 a^{n/2} \frac{\prod_{t=0}^{n/2-1} ((a-1)+bu_0u_1(a^{2t}-1))}{\prod_{t=0}^{n/2-1} ((a-1)+bu_0u_1(a^{2t+1}-1))} = u_0 a^{n/2} \frac{\prod_{t=1}^{n/2-1} (1+bu_0u_1 \sum_{i=0}^{2t-1} a^i)}{\prod_{t=0}^{n/2-1} (1+bu_0u_1 \sum_{i=0}^{2t} a^i)} & n \text{ even} \\ u_1 a^{\frac{n-1}{2}} \frac{\prod_{t=0}^{\frac{n-1}{2}-1} ((a-1)+bu_0u_1(a^{2t+1}-1))}{\prod_{t=1}^{\frac{n-1}{2}} ((a-1)+bu_0u_1(a^{2t}-1))} = u_1 a^{\frac{n-1}{2}} \frac{\prod_{t=0}^{\frac{n-1}{2}-1} (1+bu_0u_1 \sum_{i=0}^{2t} a^i)}{\prod_{t=1}^{\frac{n-1}{2}} (1+bu_0u_1 \sum_{i=0}^{2t-1} a^i)} & n \text{ odd} \end{cases} \quad (35)$$

- If we let $x_n = u_{n+1}$, $u_0 = k$, $u_1 = h$, $a = -1$ and $b = -1$, we get the result

$$\begin{cases} x_{2t+1} = \frac{k}{(hk-1)^{t+1}} \\ x_{2t+2} = h(hk-1)^{t+1} \end{cases} \quad (36)$$

obtained by C. Cinar in [8] for

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}} \quad (37)$$

and his restriction $(x_{-1}x_0 \neq 1)$ coincides with our restriction given in (34), that is, $1/(x_{-1}x_0) \notin \{1\}$ in this case.

- If we let $x_n = u_{n+1}, u_0 = k, u_1 = h, a = -1$ and $b = -B$, we get the result

$$\begin{cases} x_{2t+1} = \frac{k}{(Bhk-1)^{t+1}} \\ x_{2t+2} = h(Bhk-1)^{t+1} \end{cases} \quad (38)$$

obtained by C. Cinar in [9] for

$$x_{n+1} = \frac{x_{n-1}}{-1 + Bx_n x_{n-1}} \quad (39)$$

and his restriction $(Bx_{-1}x_0 \neq 1)$ coincides with our restriction given in (34), that is, $1/(Bx_{-1}x_0) \notin \{1\}$ in this case.

- If we let $x_n = u_{n+1}, a = -b = 1/A$, we get the result

$$x_n = \begin{cases} x_0 \prod_{i=1}^{\frac{n}{2}} \frac{A^{2i-1}(1-A) - (1-A^{2i-1})x_{-1}x_0}{A^{2i}(1-A) - (1-A^{2i})x_{-1}x_0} & n \text{ even} \\ x_{-1} \prod_{i=0}^{\frac{n+1}{2}-1} \frac{A^{2i}(1-A) - (1-A^{2i})x_{-1}x_0}{A^{2i+1}(1-A) - (1-A^{2i+1})x_{-1}x_0} & n \text{ odd} \end{cases} \quad (40)$$

obtained by Alopeili in [6] for

$$x_{n+1} = \frac{x_{n-1}}{A - x_n x_{n-1}} \quad (41)$$

and his restrictions $(A \geq 0, x_{-1}x_0 \neq A^j(1-A)(1-A^j))$ is a special case of our restriction given in (34), that is, $A/(x_{-1}x_0) \notin \{1, 1 + A^{-1}, \dots, \sum_{i=0}^{n-2} A^{-i}\}$ in this case.

- If we let $x_n = u_{n+1}, u_0 = k, u_1 = h$, we get the result

$$\begin{cases} x_{2t+1} = ka^{t+1} \frac{\prod_{i=0}^{t-1} (1+bu_0u_1 \sum_{j=0}^{2i+1} a^j)}{\prod_{i=0}^t (1+bu_0u_1 \sum_{j=0}^{2i} a^j)} \\ x_{2t+2} = ha^{t+1} \frac{\prod_{i=0}^t (1+bu_0u_1 \sum_{j=0}^{2i} a^j)}{\prod_{i=0}^t (1+bu_0u_1 \sum_{j=0}^{2i+1} a^j)} \end{cases} \quad (42)$$

obtained by C.Cinar in [11] for

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}} \quad (43)$$

and his restriction (a, b, x_{-1} and x_0 are non-negative real numbers) is a special case of our restriction given in (34), that is, $-1/(bx_{-1}x_0) \notin \{1, 1+a, \dots, \sum_{i=0}^{n-2} a^i\}$ in this case.

3 Conclusion

We have used symmetry methods for difference equations to solve equation (4). The solutions were given in (33), that is,

$$u_n = \exp \left[(-1)^{n-1} \ln |u_1| + (-1)^{n-1} \sum_{k_1=1}^{n-1} (-1)^{-k_1} \ln \left| \frac{u_0 u_1 a^{k_1}}{1 + bu_0 u_1 (\sum_{i=0}^{k_1-1} a^i)} \right| \right]$$

with $-1/(bu_0 u_1) \notin \{1, 1+a, \dots, \sum_{i=0}^{n-2} a^i\}$. Contrary to the results obtained in papers [6, 11], our solutions are ‘single’ solutions (with less restrictions on a and b). For the sake of clarification, we made some change of variables and some assumptions to show that their solutions are obtained by splitting our solutions into two categories depending on the parity of n .

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